Movement of Hot Spots of the Exterior Domain of a Ball

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1 Introduction

We consider the movement of the maximum points of the solutions of the Cauchy-Neumann problem and the Cauchy-Dirichlet of the heat equation,

$$(1.1) \qquad \begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = \phi(x) & \text{in} \quad \Omega \end{cases}$$

and

$$(1.2) \qquad \begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = \phi(x) & \text{in} \quad \Omega, \end{cases}$$

where Ω is a domain with the smooth boundary $\partial\Omega$, and $\partial_t = \partial/\partial t$, $\partial_\nu = \partial/\partial\nu$, $\nu = \nu(x)$ is the outer unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$. Let u be a solution of (1.1) or (1.2). Under suitable assumptions on the initial data ϕ, the set of the maximum points of $u(\cdot, t)$,

$$H(t) = \left\{ x \in \overline{\Omega} \,:\, u(x, t) = \max_{y \in \overline{\Omega}} u(y, t) \right\}$$

is not empty for all $t > 0$. Then we call $H(t)$ the hot spots of the solution u at the time t. In this talk we study the movement of hot spots $H(t)$ of the solution u as $t \to \infty$. We next give some known results on the movement of the hot spots $H(t)$ as $t \to \infty$.

(1) <u>Ω : bounded domains</u>

Let Ω be a bounded domain with the smooth boundary $\partial\Omega$. By the Fourier expansions of the solutions of (1.1) and (1.2), we see that, for "almost all" initial data $\phi \in L^2(\Omega)$, the hot spots of the solutions tend to the maximum points of the first nonconstant eigenfunctions of Δ (see [12]).

For the zero Neumann boundary condition, Kawohl [10] conjectured that, for any convex domains Ω, the set of the maximum points of the first nonconstant eigenfunction is a subset of $\partial\Omega$. If this conjecture is true, then the hot spots of the solution of (1.1) in a bounded convex domain Ω tend to the boundary of the domain Ω for "almost all" initial data $\phi \in L^2(\Omega)$. It is known that this conjecture holds for parallelepipeds, balls, (see [10]), and two dimensional, thin convex polygonal domain with some symmetries (see [1] and [8]). For any non-convex domain Ω, we have an example of the domain where any first nonconstant eigenfunction does not take its maximum on the boundary of the domain (see [2]).

(2) <u>Ω : unbounded domains</u>

Chavel and Karp [3] studied the heat equation $\partial_t u = \Delta u$ in several Riemannian manifolds, and obtained some asymptotic properties of solutions concerning the movement of hot spots of the solution. In particular, for the Euclidean space $\mathbf{R}^N$, they proved that, for any nonzero, nonnegative initial data $\phi \in L^\infty_c(\mathbf{R}^N)$, the hot spots $H(t)$ of the solution at each time $t > 0$ are contained in the closed convex hull of the support of ϕ, and the hot spots $H(t)$ tend to the center of mass of ϕ as $t \to \infty$. Subsequently, Jimbo and Sakaguchi [9] studied the movement of hot spots of the solution of the heat equation in the half space $\mathbf{R}^N_+$ under the Dirichlet, Neumann, and Robin boundary conditions. In particular, they proved that the hot spots $H(t)$ of the solution of (1.1) in the half space $\mathbf{R}^N_+$ with the nonzero, nonnegative initial data $\varphi \in L^\infty_c(\mathbf{R}^N_+)$ satisfies

$$H(t) \subset \partial\mathbf{R}^N_+ = \{x = (x', x_N) \in \mathbf{R}^N : x_N = 0\} \tag{1.3}$$

for all sufficiently large t. We may obtain their results for the cases $\Omega = \mathbf{R}^N$ and $\Omega = \mathbf{R}^N_+$ by using the fundamental solution of the heat equation.

Next we consider the simplest exterior domain of a compact set,

$$\Omega = \{x \in \mathbf{R}^N \, : \, |x| > 1\}.$$

Even for this simple exterior domain, it is difficult to know the sign of differential of the Neumann and Dirichlet heat kernels. So it seems difficult to stydy the movement of hot spots by using the the Neumann and Dirichlet heat kernels directly. Jimbo and Sakaguchi [9] assumed the radially symmetry of the initial data ϕ, and studied the movement of the hot spots $H(t)$ of the solutions of (1.1) and (1.2). For the Cauchy-Neumann problem (1.1), they proved that the hot spots $H(t)$ satisfies

$$H(t) \subset \partial\Omega = \partial B(0,1) \tag{1.4}$$

for all sufficiently large t. Furthermore, for the Cauchy-Dirichlet problem (1.2), they proved that there exist a constant T and a function $r = r(t) \in C^\infty([T,\infty))$ such that

$$H(t) = \{x \in \mathbf{R}^N : |x| = r(t)\}, \qquad \lim_{t\to\infty} t^{-1} r(t)^3 = 2 \tag{1.5}$$

if $N = 3$. Their proofs of (1.4) and (1.5) heavily depend on the properties of zero sets of the heat equation in $\mathbf{R}$, and it seems so difficult to apply their proofs to the solutions without the radially symmetry.

In this talk we study the movement of hot spots of the solutions of (1.1) and (1.2) in the exterior domain of a ball as $t \to \infty$, without the radially symmetry of the initial data ϕ.

2 The Cauchy-Neumann Problem

In this section we consider the Cauchy-Neumann problem (1.1), and study the movement of hot spots $H(t)$ of the solution of (1.1) in the exterior domain Ω of a ball. Throughout this section we assume that

$$\Omega = \{x \in \mathbf{R}^N : |x| > L\}, \quad \phi \in L^2(\Omega, \rho dx), \quad \int_\Omega \phi(x)dx > 0, \tag{2.1}$$

where $L > 0$ and $\rho(x) = e^{|x|^2/4}$. We first give a sufficient condition for the hot spots $H(t)$ to exist only on the boundary $\partial\Omega$ for all sufficiently large t.

Theorem 2.1 *Let u be a solution of the Cauchy-Neumann problem* (1.1) *under the condition* (2.1). *Put*

$$A_N(\phi) = \int_\Omega x\phi(x)\left(1 + \frac{L^N}{N-1}|x|^{-N}\right)dx \Big/ \int_\Omega \phi(x)dx.$$

Assume

$$A_N(\phi) \in B(0,L) = \mathbf{R}^N \setminus \overline{\Omega}. \tag{2.2}$$

Then there exists a positive constant T such that

$$H(t) \subset \partial\Omega = \left\{x \in \mathbf{R}^N : |x| = L\right\} \tag{2.3}$$

for all $t \geq T$.

In particular, we see that, under the condition (2.1), the hot spots $H(t)$ of the radial solution of (1.1) exists only on the boundary of the domain Ω for all sufficiently large t.

Remark 2.1 *Let u be a solution of the Cauchy-Neumann problem* (1.1) *under the condition* (2.1). *Let $C(u(t))$ a center of mass of $u(t)$, that is,*

$$C(u(t)) = \int_\Omega xu(x,t)dx \Big/ \int_\Omega u(x,t)dx.$$

Then it does not necessarily hold that $C(u(t)) = C(\phi)$ for all $t > 0$. On the other hand, we put

$$A_N(u(t)) \equiv \int_\Omega xu(x,t)\left(1 + \frac{L^N}{N-1}|x|^{-N}\right)dx \Big/ \int_\Omega u(x,t)dx, \quad t > 0.$$

Then we have $A_N(u(t)) = A_N(\phi)$ for all $t > 0$, and $\lim_{t\to\infty} C(u(t)) = A(\phi)$.

Next we give a result on the limit set of $H(t)$ as $t \to \infty$.

Theorem 2.2 *Let u be a solution of the Cauchy-Neumann problem* (1.1) *under the condition* (2.1). *Assume $A_N(\phi) \neq 0$. Put*

$$x_\infty = L\frac{A_N(\phi)}{|A_N(\phi)|} \quad \textit{if} \quad A_N(\phi) \in B(0,L) \quad \textit{and} \quad x_\infty = A_N(\phi) \quad \textit{if} \quad A_N(\phi) \in \overline{\Omega}$$

Then

$$\lim_{t\to\infty} \sup\left\{|x_\infty - y| : y \in H(t)\right\} = 0.$$

By Theorem 2.2, we see that the hot spots $H(t)$ tends to one point x_∞ as $t \to \infty$ if $A_N(\phi) \neq 0$, and see that (2.3) does not hold if $A_N(\phi) \in \Omega$ (compare with (1.3) and (1.4)).

3 The Cauchy-Dirichlet Problem

In this section we consider the Cauchy-Dirichlet problem (1.2), and study the movement of hot spots $H(t)$ of the solution of (1.2) in the exterior domain Ω of a ball. Throughout this section we assume that

$$\Omega = \{x \in \mathbf{R}^N : |x| > L\}, \quad \phi \in L^2(\Omega, e^{|x|^2/4}dx), \qquad m_\phi > 0, \tag{3.1}$$

where $L > 0$ and

$$m_\phi = \begin{cases} \displaystyle\int_\Omega \phi(x)\left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right)dx & \text{if } N \geq 3, \\ \displaystyle\int_\Omega \phi(x)\log\frac{|x|}{L}dx & \text{if } N \geq 2. \end{cases}$$

We first give the following theorems on the asymptotic behavior of the solution u of (1.2), which implies that the hot spots $H(t)$ run away from the boundary $\partial\Omega$ as $t \to \infty$.

Theorem 3.1 *Let u be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1) *and $N \geq 3$. Then*

$$\lim_{t\to\infty} \int_\Omega u(x,t)dx = m_\phi > 0 \tag{3.2}$$

and

$$\lim_{t\to\infty} t^{\frac{N}{2}} u(x,t) = (4\pi)^{-\frac{N}{2}} m_\phi \left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right) \tag{3.3}$$

uniformly for all x on any compact set in $\overline{\Omega}$.

Theorem 3.2 *Let u be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1) *and $N = 2$. Then there exists a constant C such that*

$$\|u(\cdot,t)\|_{L^1(\Omega)} \leq C(\log t)^{-1}\|\phi\|_{L^2(\Omega,\rho dx)} \tag{3.4}$$

for all $t \geq 1$. Furthermore

$$\lim_{t\to\infty} (\log t) \int_\Omega u(x,t)dx = 2m_\phi \tag{3.5}$$

and

$$\lim_{t\to\infty} t(\log t)^2\, u(x,t) = \frac{1}{\pi} m_\phi \log\frac{|x|}{L} \tag{3.6}$$

uniformly for all x on any compact set in $\overline{\Omega}$.

Remark 3.1 *Collet, Martínes, and Martín* [4] *proved the asymptotic behavior of the Dirichlet heat kernel* $G = G(x, y, t)$ *on the exterior domain of a compact set as* $t \to \infty$. *In particular, for the exterior domain* $\mathbf{R}^N \setminus \overline{B(0, L)}$, *they obtained that*

$$(3.7) \quad \lim_{t\to\infty} t^{\frac{N}{2}} G(x, y, t) = (4\pi)^{-\frac{N}{2}} \left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right)\left(1 - \frac{L^{N-2}}{|y|^{N-2}}\right) \quad \text{if } N \geq 3,$$

$$(3.8) \quad \lim_{t\to\infty} t(\log t)^2 G(x, y, t) = \frac{1}{\pi} \log \frac{|x|}{L} \log \frac{|y|}{L} \qquad \text{if } N = 2,$$

for all $x, y \in \Omega$ *(see also* [6]*). By* (3.3) *and* (3.6)*, we may obtain* (3.7) *and* (3.8)*, and the proof of this paper is complete different from the one of* [4]*. Furthermore we remark that Herraiz* [7] *applied the comparison method to the Cauchy-Dirichlet problem* (1.2) *in general exterior domains and obtained the similar results to Theorems* 3.1 *and* 3.2 *for nonnegative initial data* ϕ.

Next we give a result on the rate for the hot spots $H(t)$ to run away from the boundary Ω as $t \to \infty$.

Theorem 3.3 *Let* u *be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1)*. Put*

$$\zeta(t) = 2(N-2)L^{N-2}t \quad \text{if } N \geq 3, \qquad \zeta(t) = 2t(\log t)^{-1} \quad \text{if } N = 2.$$

Then

$$(3.9) \qquad \lim_{t\to\infty} \sup_{x\in H(t)} \left| \zeta(t)^{-1}|x|^N - 1 \right| = 0.$$

Furthermore there exists a positive constant T *such that, if* $x \in H(t)$ *and* $t \geq T$*, then*

$$(3.10) \qquad H(t) \cap l_x = \{x\},$$

where $l_x = \{x \in \mathbf{R}^N \,:\, kx/|x|,\ k \geq 0\}$.

Remark 3.2 *Let* u *be a radial solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1)*. Then we see that there exists a smooth curve* $r = r(t) \in (L, \infty)$ *such that* $H(t) = \{x \in \mathbf{R}^M \,:\, |x| = r(t)\}$ *for all sufficiently large* t.

Next we give a sufficient condition for the hot spots $H(t)$ to consists of one point $x(t)$ after a finite time. Furthermore we give the limit of $x(t)/|x(t)|$ as $t \to \infty$.

Theorem 3.4 *Let u be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1). *Assume that*

$$A_D(\phi) \equiv \int_\Omega x\phi(x)\left(1 - \frac{L^N}{|x|^N}\right) dx \neq 0.$$

Then there exist a positive constant T and a smooth curve $x = x(t) \in C^\infty([T,\infty) : \Omega)$ such that $H(t) = \{x(t)\}$ for all $t \geq T$ and

$$\lim_{t\to\infty} \frac{x(t)}{|x(t)|} = \frac{A_D(\phi)}{|A_D(\phi)|}. \tag{3.11}$$

4 Outline of the Proofs

In this section we give the outline of the proofs of Theorems 2.1 and 2.2 only. In order to prove Theorems 2.1 and 2.2, we consider the asymptotic behavior of the radial solution v_k of the Cauhy-Neumann problem (L_k):

$$(L_k) \qquad \begin{cases} \partial_t v_k = \mathcal{L}_k v_k \equiv \Delta v_k - \dfrac{\omega_k}{|x|^2} v_k & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu v_k = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ v_k(x,0) = \phi_k(x) & \text{in} \quad \Omega, \end{cases}$$

where $k \in \mathbf{N} \cup \{0\}$ and ϕ_k is a radial function belonging to $L^2(\Omega, \rho dx)$ with $\rho(y) = \exp(|y|^2/4)$. Here $\{\omega_k\}_{k=0}^\infty$ be the eigenvalues of

$$-\Delta_{\mathbf{S}^{N-1}} Q = \omega Q \qquad \text{on} \quad \mathbf{S}^{N-1}, \tag{4.1}$$

such that $0 = \omega_0 < \omega_1 = N-1 < \omega_2 = 2N < \omega_3 < \cdots$, where $\Delta_{\mathbf{S}^{N-1}}$ is the Laplace-Beltrami operator on $\mathbf{S}^{N-1}$. Furthermore we define a rescaled function w_k of the solution v_k as follows:

$$w_k(y,s) = (1+t)^{\frac{N+k}{2}} v_k(x,t), \quad y = (1+t)^{-\frac{1}{2}} x, \quad s = \log(1+t). \tag{4.2}$$

Then the function w_k satisfies

$$(P_k) \qquad \begin{cases} \partial_s w_k = P_k w_k + \dfrac{N+k}{2} w_k & \text{in} \quad W, \\ \partial_\nu w_k = 0 & \text{on} \quad \partial W, \\ w_k(y,0) = \phi(y) & \text{in} \quad \Omega, \end{cases}$$

where

$$
\begin{aligned}
&P_k w = \Delta_y w + \frac{y}{2} \cdot \nabla_y w - \frac{\omega_k}{|y|^2} w = \frac{1}{\rho} \mathrm{div}\,(\rho \nabla_y w) - \frac{\omega_k}{|y|^2} w, \\
&\Omega(s) = e^{-s/2}\Omega, \ W = \bigcup_{0<s<\infty} (\Omega(s) \times \{s\}), \ \partial W = \bigcup_{0<s<\infty} (\partial\Omega(s) \times \{s\}).
\end{aligned}
$$

We study the asymptotic behavior of the first eigenvalue and the first eigenfunction of the operator P_k, and obtain the asymptotic behavior of the solution w_k in the space L^2 with weight ρ. Furthermore, for $k = 0, 1, 2$, by using the radially symmetry of v_k, the equations (L_k) and (P_k), and the Ascoli-Arzera theorem, we study the asymptotic behavior of v_k, $\partial_r v_k$, and $\partial_r^2 v_k$ as $t \to \infty$. Finally we study the asymptotic behavior of u, ∇u, and $\nabla^2 u$ as $t \to \infty$ by using the results on v_k, $\partial_r v_k$, and $\partial_r^2 v_k$, and prove Theorems 2.1 and 2.2.

For the case $k = 0$, we extend the domain of w_0 to $\mathbf{R}^N$, and apply the Ascoli-Arzera theorem to w_0. Then, by using the results on the asymptotic behavior of w_0 in the space L^2 with weight ρ, we obtain a result on the asymptotic behavior of v_0 and $\partial_r v_0$, where $r = |x|$. Furthermore we obtain a result on the asymptotic behavior of $\partial_r^2 v_0$ as $t \to \infty$ by using the ones of v_0 and $\partial_r v_0$. On the other hand, for the case $k = 1$, the inequality

$$
\sup_{s>1} \|\nabla_y^2 w_1(\cdot, s)\|_{C(\Omega(s))} < \infty
$$

does not necessarily holds, and $w(y, s)$ tends to 0 uniformly for all y with $|y| \le Re^{-s/2}$ with any $R > L$. So it is not useful to apply the Ascoli-Arzera theorem to w_1 for the aim at studying the asymptotic behavior of w_1 and $\partial_r w_1$ in the domain $\{y \in \Omega(s) : |y| \le Re^{-s/2}\}$, as $s \to \infty$. To overcome this difficulty, we may apply the Ascoli-Arzera theorem w_1 in the any annulus $D(\epsilon, R) = \{y \in \mathbf{R}^N : \epsilon \le |y| \le R\}$ with $0 < \epsilon < R$, and obtain the asymptotic behavior of w_1 in the annulus $D(\epsilon, R)$. Furthermore we use the equation (L_1) effectively, and study the asymptotic behavior of v_1, $\partial_r v_1$ and $\partial_r^2 v_1$ as $t \to \infty$. For $k = 2$, we apply the similar arguments to in w_1 to w_2, and study the asymptotic behavior of v_2, $\partial_r v_2$ and $\partial_r^2 v_2$ as $t \to \infty$.

For the Cauchy-Dirichlet problem (1.2), we follow the strategy for the proofs of Theorems 2.1 and 2.2, and study the asymptotic behavior of the solutions of (1.2) to prove Theorems 3.1–3.4.

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