On small amplitude global solutions for the nonlinear Klein-Gordon equation

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This talk is based on the joint work with Tohru Ozawa (Hokkaido University). In this talk, we consider the Cauchy problem of the quadratic nonlinear Klein-Gordon equation in two space dimensions,

\[ \partial_t^2 u - \Delta u + u = Q(u, \partial u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \]

\[ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R}, \]

where \( \partial = (\partial_t, \partial_1, \partial_2) \), \( Q \) is the quadratic nonlinearity in \( u \) and \( \partial u \).

There are many studies concerning the global existence and asymptotic behavior of solutions for nonlinear Klein-Gordon equations. Let \( n \) be the spatial dimensions. When \( n \geq 5 \), Klainerman-Ponce [5] and Shatah [8] showed that the Cauchy problem (1)-(2) has the unique global solution for small initial data and the solution asymptotically approaches to the free solution of the linear Klein-Gordon equation as \( t \to \infty \). The proofs in [5] and [8] are based on the \( L^p - L^q \) estimate of the solution to the linear Klein-Gordon equation.

When \( n \leq 4 \), the \( L^p - L^q \) estimate does not provide us a sufficient time decay to construct global solutions. To overcome this difficulty, Klainerman [4] introduced the invariant Sobolev space with respect to the generators of the Lorentz group and showed the existence of global solution to (1)-(2) when \( n = 3, 4 \). Independently, Shatah [9] introduced the method of the normal forms, which is the extension of the Poincaré’s theory of normal forms for the ordinary differential equations to the nonlinear Klein-Gordon equations, and showed the existence of global solution to (1)-(2) when \( n = 3, 4 \).

When \( n = 2 \), Georgiev-Popivanov [2] and Kosecki [3] showed the existence of global solution provided that the nonlinearity in (1) is the special form. The general nonlinearities are treated by Simon-Taflin [10] and Ozawa-Tsutaya-Tsutsumi [7]. In particular, the proof in [7] is based on the method of normal forms and the decay estimate due to Georgiev [1], and requires the following conditions on the initial data other than the smallness,

\[ u_0 \in H^{k+16,k+15}(\mathbb{R}^2), \quad u_1 \in H^{k+15,k+15}(\mathbb{R}^2), \quad k \geq 21, \]

where \( H^{m,s} \) denotes the weighted Sobolev space whose norm is defined by

\[ \| f \|_{H^{m,s}} = \|(1 + |x|^2)^{s/2}(1 - \Delta)^{m/2} f \|_{L^2}. \]
The purpose of this talk is to give a simple proof and to relax the condition of the initial data for the existence of global solutions to (1)-(2) by using the endpoint Strichartz estimates in mixed norms on the polar coordinates. Such estimates for the wave and the Klein-Gordon equation in three space dimensions are introduced by Machihara-Nakamura-Nakanishi-Ozawa in [6].

References


