LEGENDRIAN SINGULARITIES AND FIRST ORDER DIFFERENTIAL EQUATIONS

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LEGENDRIAN SINGULARITIES AND FIRST ORDER DIFFERENTIAL EQUATIONS

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1.INTRODUCTION

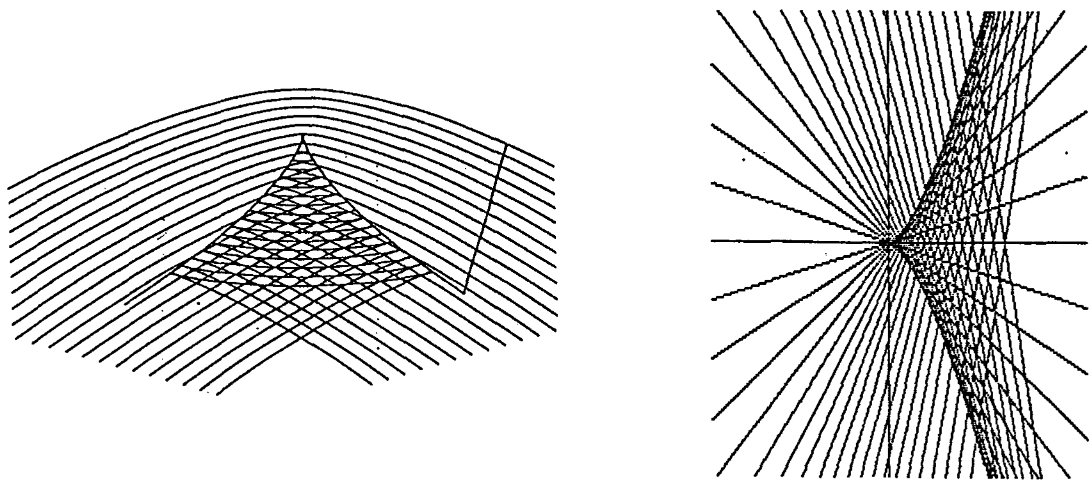
This is a summary of the papers [8,9,10], which mainly study generic properties of first order differential equations in some sense.

In the first place we consider the following two example of ordinary differential equations:

$$(1)\ (\frac{dy}{dx})^3 - y\frac{dy}{dx} - x = 0.$$

$$(2)\ (\frac{dy}{dx})^3 - x\frac{dy}{dx} - y = 0.$$

These equations have first integrals and we can draw these pictures of graphs of solutions by a computer as follows :



(1) (2)

Fig. 1

Here we can interpret the picture (1) is the projection image of curves which are given by cutting the swallowtail by generic surfaces (Fig. 2).

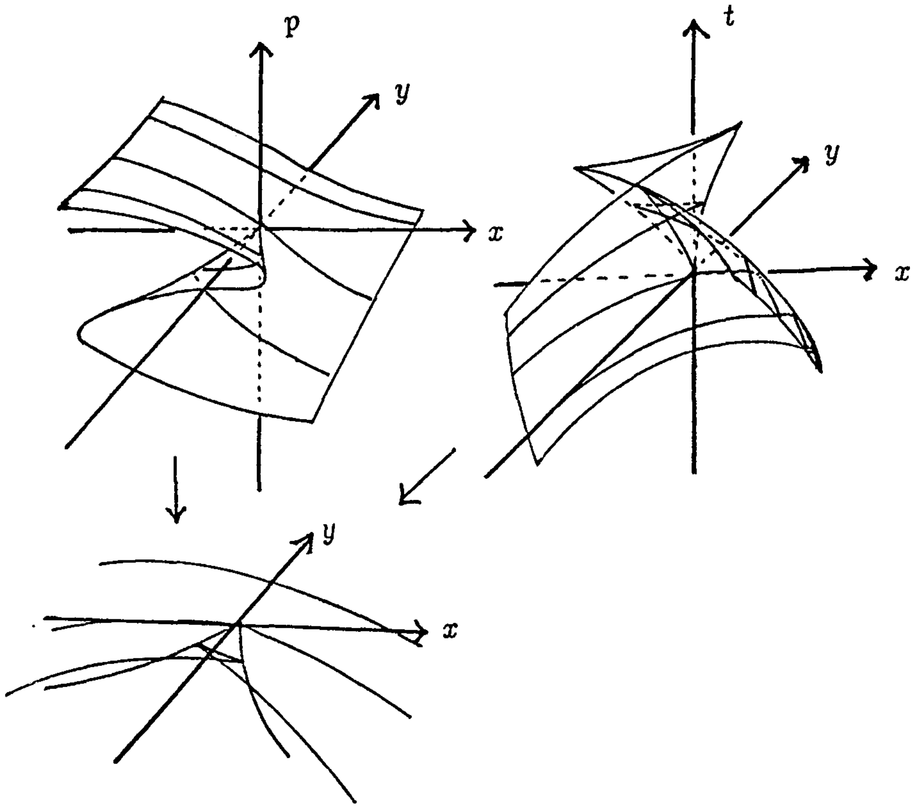


Fig. 2

Then there is a natural question : why does this situation happen ? According to this question, there are many studies from a long time ago [3,4,6,7,11]. In these studies it has been known that the discriminant set and solutions are cuspidal curves for generic ordinary differential equations $F(x, y, \frac{dy}{dx}) = 0$ and the Clairaut's equation such as (2) is not generic.

On the other hand, there are very few studies for partial differential equations. Probably there were no tools to study these objects in those times.

We now review the classical theory of first order partial differential equations : In the classical theory *a first order partial differential equation* (*or, briefly, an equation*) is written in the form

$$F_k(x_1, \ldots, x_n, y, p_1, \ldots, p_n) = 0 \ (k = 1, \ldots, 2n + 1 - r, r \geq n).$$

A (*classical*) *solution* of the equation is a smooth function

$$y = f(x_1, \ldots, x_n) \text{ and } p_i = \frac{\partial f}{\partial x_i}(x).$$

We usually assume that F_k are $(2n + 1)$-variable smooth function and

$$\operatorname{rank}(\frac{\partial F_k}{\partial x_i}, \frac{\partial F_k}{\partial y}, \frac{\partial F_k}{\partial p_j}) = 2n = 1 - r.$$

Define

$$\begin{aligned} D =&\{(x,y)|\text{there exists } p\in \mathbf{R}^n \text{such that}\\ &F_1(x,y,p)=\cdots=F_{2n+1}(x,y,r)=0\\ &\text{and } \operatorname{rank}(\frac{\partial F_k}{\partial p_j})(x,y,p)<\min(n,2n=1-r)\}. \end{aligned}$$

We call D *a discriminant set* of the equation. We also define

$$\begin{aligned} \Sigma =&\{(x,y,p)|F_1(x,y,p)=\cdots=F_{2n+1-r}(x,y,p)=0\\ &\text{and } \operatorname{rank}(\frac{\partial F_k}{\partial p_j})(x,y,p)<\min(n,2n+1-r)\}. \end{aligned}$$

We say that Σ is *a singular solution of the equation* if D is a graph of a solution. In the history of differential equations, the notion of a singular solution appeared with the notion of a complete solution (cf. [2]). We say that $(r-n)$-parameter family of (classical) solutions $y=f(t_1,\dots,t_{r-n},x_1,\dots,x_n)$ of the equation is *a (classical) complete solution* if

$$\operatorname{rank}(\frac{\partial f}{\partial t_j},\frac{\partial^2 f}{\partial t_i\partial x_j})=r-n.$$

The following theorem is one of the best results in the classical theory.

THEOREM 1.1. (CLASSICAL EXISTENCE THEOREM). *If the equation is involutory near a point (x_0,y_0,p_0) and $(x_0,y_0,p_0)\notin\Sigma$, then there exists a (classical) complete solution of the equation near (x_0,y_0,p_0).*

We say that the equation is involutory if $[F_j,F_k]=0$ for $j,k=1,\dots,2n+1-r$, where

$$\begin{aligned} [F,G]=F\cdot\frac{\partial G}{\partial z}-G\cdot\frac{\partial F}{\partial z}+\sum_{i=1}^{n}(\frac{\partial F}{\partial x_i}\cdot\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\cdot\frac{\partial G}{\partial x_i})\\ +\sum_{i=1}^{n}p_i\cdot(\frac{\partial F}{\partial z}\cdot\frac{\partial G}{\partial p_i}-\frac{\partial G}{\partial z}\cdot\frac{\partial F}{\partial p_i}). \end{aligned}$$

This result leads us to ask : What happens at a point of Σ? In relation to this question, we have the following two examples.

EXAMPLE 1.2. We consider the following first order partial differential equations of two independent variables.

$$(1)\qquad \begin{cases} (\frac{\partial y}{\partial x_1})^2-x_1=0\\ \frac{\partial y}{\partial x_2}=0, \end{cases}$$

$$(2)\qquad \begin{cases} y-(\frac{\partial y}{\partial x_1})^2=0\\ \frac{\partial y}{\partial x_2}=0. \end{cases}$$

We can exactly solve these equations. Then complete solutions of these equations are given by

$$(1)'\ y = \pm\frac{2}{3}x_1^{\frac{2}{3}} + t,$$

$$(2)'\ y = \frac{1}{4}(x_1 + t)^2.$$

We can understand that the solution of the equation (1) has multivalued near the π-singular set Σ and this equation does not have the singular solution. On the other hand, the equation (2) has a classical complete solution near Σ and the singular solution. Moreover, Σ is an envelope of the solution $(2)'$.

2. Geometry of first order differential equations

The aim of this section is to describe the geometric structure connected with first order differential equations. Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of functions of n-variables. Since we only consider the local situation, the 1-jet bundle $J^1(\mathbb{R}^n, \mathbb{R})$ may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system

$$(x_1, \ldots, x_n, y, p_1, \ldots, p_n)$$

where $(x_1, \ldots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We have the canonical projection $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ by $\pi(x, y, p) = (x, y)$. Let θ be the canonical 1-form on $J^1(\mathbb{R}^n, \mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$. Throughout the remainder of this note, we shall consider $J^1(\mathbb{R}^n, \mathbb{R})$ as a contact manifold whose contact structure is given by the canonical 1-form. Using this approach, a first order differential equation is most naturally interpreted as being a closed subset of $J^1(\mathbb{R}^n, \mathbb{R})$. Unless the contrary is specially stated, we use the following definition. *A first order differential equation (or, briefly, an equation)* is an r-dimensional submanifold $E \subset J^1(\mathbb{R}^n, \mathbb{R})$ where $n + 1 \leq r \leq 2n$. If $r < 2n$, then E is said to be *overdetermined.* We also say that E is *maximally overdetermined (or, holonomic)* if $r = n + 1$. The notion of a solution of an equation is given by the philosophy of Lie. *An (abstract) solution* of E is a Legendrian immersion such that $i(L) \subset E$. Here, i is a Legendrian immersion if $\dim L = n$ and $i^*\theta = 0$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then the jet extension $j^1 f : \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R})$ is a Legendrian embedding. Hence, in our terminology, the (classical) solution of E is a smooth function f such that $j^1 f(\mathbb{R}^n) \subset E$. On the other hand, we can show that an (abstract) solution i is given by (at least locally) a jet extension of a smooth function if and only if $\pi \circ i$ is a non-singular map. Thus we can define the notion of singularities of solutions. We say that $z_0 \in L$ is *a Legendrian singular point* if z_0 is a singular point of $\pi \circ i$. We also define notions of singularities of an equation E. We say that $z_0 \in E$ is *a contact singular point* if $\theta(T_{z_0} E) = 0$. We also say that $z_0 \in E$ is *a π-singular point* if z_0 is a singular point of $\pi|E$. Let $\Sigma(E)$ be the set of π-singular points. We call $D_E = \pi(\Sigma(E))$ *a discriminant set* of the equation E. The notion of a singular solution is as follows. If the π-singular set $\Sigma(E)$ is a Legendrian submanifold, then we call it *a singular solution* of E. In [9] we have the following theorem in the case when $\dim E = 2n$ (i.e. the single equation case).

THEOREM 2.1 ([9], THEOREM 2.2). *Generic equation germs (E, z_0) in the set of all single equation germs have no singular solution and the point z_0 satisfies one of the following conditions :*

(1) *z_0 is a π-regular point.*

(2) *z_0 is a fold type singular point of $\pi|E$ and an isolated contact singular point.*

(3) *z_0 is a π-singular point and a contact regular point.*

We have some examples which suggest the above result.

EXAMPLE 2.2. Consider the case when $n = 2$. The followings are defined near the origin $0 \in J^1(\mathbf{R}^n, \mathbf{R})$.

(1) The origin is a fold point of $\pi|F^{-1}(0)$ and an isolated contact singular point.

$$F(x_1, x_2, y, p_1, p_2) = p_1^2 + p_2^2 + x_1^2 + x_2^2 - y$$

(2) The origin is a fold point of $\pi|F^{-1}(0)$ and a contact regular point.

$$F(x_1, x_2, y, p_1, p_2) = p_1^2 + p_2^2 + x_1 + x_2 p_2 + y p_1$$

(3) The origin is a cusp point of $\pi|F^{-1}(0)$ and a contact regular point.

$$F(x_1, x_2, y, p_1, p_2) = p_1^3 + p_2^2 + y p_1 + x_1$$

(4) The origin is a swallowtail point of $\pi|F^{-1}(0)$ and a contact regular point.

$$F(x_1, x_2, y, p_1, p_2) = p_1^4 + p_2^2 + y p_1^2 + x_1 p_1 + x_2$$

On the other hand, the Clairaut's equation is non-generic.

EXAMPLE 2.3. The Clairaut's equation : Consider the following partial differential equation on $J^1(\mathbf{R}^n, \mathbf{R})$.

$$y = x_1 p_1 + \cdots + x_n p_n + f(p_1, \ldots, p_n)$$

The (classical) complete solution of this equation is given by

$$y = x_1 t_1 + \cdots + x_n t_n + f(t_1, \ldots, t_n),$$

where $(t_1, \ldots, t_n)$ are parameters. The envelope of this family is the singular solution of the equation.

In [9] we have proved the following theorem suggested by the above example.

THEOREM 2.4 ([9], COROLLARY 3.3). *Let (E, z_0) be an equation with singular solution. Suppose that the set of fold points of $\pi|E$ are dense in $\Sigma(E)$. Then this equation has a (classical) complete solution and $\Sigma(E)$ is an envelope of such a family of solutions.*

These theorems are generalizations of some results in ([3],[11]). But it seems to be very hard to generalize these results in the case when $\dim E < 2n$.

On the other hand, the notion of a (classical) complete solution is very important in the classical theory as in Theorem 1.1. Let $y = f(t_1, \dots, t_n, x_1, \dots, x_n)$ be the (classical) complete solution of E. Then the condition $\mathrm{rank}(\frac{\partial f}{\partial t_j}, \frac{\partial^2 f}{\partial t_i \partial x_j}) = r - n$ say that the map germ

$$j^1 f_* : (\mathbf{R}^{r-n} \times \mathbf{R}^n, 0) \to J^1(\mathbf{R}^n, \mathbf{R})$$

defined by $j^1 f_*(t, x) = j^1 f_t(x)$ is an immersion. Then $j^1 f_*$ gives a local parametrization of E. Thus we have the following definition. We say that an equation $E \subset J^1(\mathbf{R}^n, \mathbf{R})$ is *completely integrable (or E has an (abstract) complete solution)* if there exists an n-dimensional completely integrable distribution $\mathcal{D}$ on E such that

$$\theta_z(\mathcal{D}_z) = 0$$

for any $z \in E$.

By the Frobenius' theorem, we have the following proposition.

PROPOSITION 2.5. *Let $E^r \subset J^1(\mathbf{R}^n, \mathbf{R})$ be an equation. Then the following conditions are equivalent :*

(1) *E is completely integrable.*

(2) *For any $z \in E$, there exists a neighbourhood U of z in E and smooth functions*

$$\mu_1, \dots, \mu_{r-n} \text{ on } U$$

such that

$$d\mu_1 \wedge \cdots \wedge d\mu_{r-n} \neq 0 \text{ on } U$$

and

$$\langle d\mu_1, \dots, d\mu_{r-n} \rangle_{C^\infty(U)} \supset \langle \theta | U \rangle_{C^\infty(U)}$$

as $C^\infty(U)$-modules, where $C^\infty(U)$ denotes the ring of smooth functions on U.

(3) *For any $z \in E$, there exist a neighbourhood $V \times W$ of 0 in $\mathbf{R}^{r-n} \times \mathbf{R}^n$ and an embedding*

$$f : V \times W \to J^1(\mathbf{R}^n, \mathbf{R})$$

such that

$$f(0) = q, \; f(V \times W) \subset E$$

and $f|t \times W$ are Legendrian embeddings for any $t \in V$.

3. Completely integrable first order differential equations

Since we will only study local properties, an equation is defined to be an immersion

$$f : U \to J^1(\mathbf{R}^n, \mathbf{R})$$

where U is an open subset of $\mathbf{R}^n$. By Proposition 2.5, we say that f is *completely integrable* if there exists a submersion

$$\mu = (\mu_1, \dots, \mu_{r-n}) : U \to \mathbf{R}^{r-n}$$

such that

$$\langle d\mu_1, \ldots, \mu_{r-n} \rangle_{C^\infty(U)} \supset \langle f^*\theta \rangle_{C^\infty(U)}.$$

We call μ *a complete integral* of f and the pair

$$(\mu, f) : U \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$$

is called *an equation with complete integral.*

We can also define the above notions in terms of map germs : *An equation germ* is defined to be an immersion germ

$$f : (\mathbf{R}^r, 0) \to J^1(\mathbf{R}^n, \mathbf{R}).$$

We say that f is *completely integrable* if there exists a submersion germ

$$\mu = (\mu_1, \ldots, \mu_{r-n}) : (\mathbf{R}^r, 0) \to \mathbf{R}^{r-n}$$

such that

$$\langle d\mu_1, \ldots, \mu_{r-n} \rangle_{\mathcal{E}_u} \supset \langle f^*\theta \rangle_{\mathcal{E}_u},$$

where $u = (u_1, \ldots, u_r)$ is the canonical coordinate of $(\mathbf{R}^r, 0)$ and $\mathcal{E}_u$ denotes the ring of smooth function germs of u-variables at the origin. Then μ is called *a complete integral* of f and the pair

$$(\mu, f) : (\mathbf{R}^r, 0) \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$$

is called *an equation germ with complete integral.* This situation leads us to the following definition. Let (μ, g) be a pair of a map germ

$$g : (\mathbf{R}^r, 0) \to (\mathbf{R}^n \times \mathbf{R}, 0)$$

and a submersion germ

$$\mu : (\mathbf{R}^r, 0) \to (\mathbf{R}^{r-n}, 0).$$

Then the diagram

$$(\mathbf{R}^{r-n}, 0) \xleftarrow{\mu} (\mathbf{R}^r, 0) \xrightarrow{g} (\mathbf{R}^n \times \mathbf{R}, 0)$$

or briefly (μ, g), is called *an integral diagram* if there exists an equation germ

$$f : (\mathbf{R}^r, 0) \to J^1(\mathbf{R}^n, \mathbf{R})$$

such that (μ, f) is an equation with complete integral and $\pi \circ f = g$, and we say that (μ, g) *is induced by* f. In the case when $r = n + 1$, an equation germ $f : (\mathbf{R}^{n+1}, 0) \to J^1(\mathbf{R}^n, \mathbf{R})$ is called *a holonomic equation germ* and an integral diagram

$$(\mathbf{R}, 0) \xleftarrow{\mu} (\mathbf{R}^{n+1}, 0) \xrightarrow{g} (\mathbf{R}^n \times \mathbf{R}, 0)$$

is called *a holonomic integral diagram.* By the definition, we have the following simple lemma.

LEMMA 3.1. *Let (μ, f) be an equation with complete integral. Then there exist unique elements $h_1, \ldots, h_{r-n} \in C^\infty(U)$ such that*

$$f^*\theta = \sum_{i=1}^{r-n} h_i d\mu_i \text{ on } U.$$

We now consider the 1-jet bundle $J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$ and the canonical 1-form Θ on the space. Let

$$(t_1, \ldots, t_{r-n}, x_1, \ldots, x_n, y, q_1, \ldots, q_{r-n}, p_1, \ldots p_n)$$

be the coordinate system on $J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$ which corresponds to coordinate system $(t_1, \ldots, t_{r-n}, x_1, \ldots, x_n)$ on $\mathbf{R}^{r-n} \times \mathbf{R}^n$. Then the canonical 1-form is given by

$$\Theta = dy - \sum_{j=1}^{r-n} q_j dt_j - \sum_{i=1}^{n} p_i dx_i.$$

We also define the canonical projection

$$\Pi : J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}) \to \mathbf{R}^{r-n} \times \mathbf{R}^n \times \mathbf{R}$$

by

$$\Pi(t, x, y, q, p) = (t, x, y).$$

We call this 1-jet bundle *an unfolded 1-jet bundle.*

Let (μ, f) be an equation with complete integral. By Lemma 3.1, we can define a mapping

$$\ell_{(\mu,f)} : U \to J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$$

by

$$\ell_{(\mu,f)}(u) = (\mu(u), x \circ f(u), y \circ f(u), h(u), p \circ f(u)).$$

By Lemma 3.1, $\ell_{(\mu,f)}$ is a Legendrian immersion. We call it *a Legendrian unfolding associated with* (μ, f).

Our purpose is to study the genericity of some properties of equation germs with complete integral. Let $U \subset \mathbf{R}^r$ be an open set. We denote

$$\mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}))$$

the set of equations with complete integral

$$(\mu, f) : U \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}).$$

We also denote

$$\mathrm{L}(U, J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}))$$

to be the set of Legendrian unfoldings

$$\ell_{(\mu,f)} : U \to J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}).$$

These sets are topological spaces equipped with the Whitney C^∞-topology. Then we can define a mapping

$$\mathcal{L} : \mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})) \to \mathrm{L}(U, J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}))$$

by

$$\mathcal{L}(\mu, f) = \ell_{(\mu,f)}.$$

The following theorem is the base of our theory.

THEOREM 3.2 ([8,10]). *The mapping $\mathcal{L}$ is a homeomorphism.*

Now we will consider the genericity of equations with complete integral as follows. A subset of $\mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}))$ is called *generic* if it is an open and dense subset of $\mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}))$. We say that *generic equation germ with complete integral* $(\mu, f) : (\mathbf{R}^r, 0) \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$ has a property P if, for some neighbourhood U of 0 in $\mathbf{R}^r$, the set

$$\begin{aligned} P(U) = \{(\mu', f') \in \mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})) | \\ \text{the germ } (\mu', f') : (U, a) \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}) \\ \text{has the property } P \text{ for any point } a \in U\} \end{aligned}$$

is a generic set in $\mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}))$. Theorem 3.2 asserts that the genericity of a property of equations with complete integral can be interpreted by the genericity of the corresponding property of Legendrian unfoldings. On the other hand, by the theory of Arnol'd-Zakalyukin [1,12], we can study generic properties of Legendrian unfoldings in terms of generating families. Let

$$F : ((\mathbf{R}^{r-n} \times \mathbf{R}^n) \times \mathbf{R}^k, 0) \to (\mathbf{R}, 0)$$

be a function germ such that

$$d_2 F | 0 \times \mathbf{R}^n \times \mathbf{R}^k$$

is non-singular, where

$$d_2 F(t, x, q) = (\frac{\partial F}{\partial q_1}(t, x, q), \dots, \frac{\partial F}{\partial q_k}(t, x, q)).$$

Then $C(F) = d_2 F^{-1}(0)$ is a smooth r-manifold germ and $\pi_F : (C(F), 0) \to \mathbf{R}^{r-n}$ is a submersion germ, where $\pi_F(t, x, q) = t$.

Define map germs

$$\tilde{\Phi}_F : (C(F), 0) \to J^1(\mathbf{R}^n, \mathbf{R})$$

by

$$\tilde{\Phi}_F(t, x, q) = (x, F(t, x, q), \frac{\partial F}{\partial x}(t, x, q))$$

and

$$\Phi_F : (C(F), 0) \to J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$$

by

$$\Phi_F(t, x, q) = (t, x, F(t, x, q), \frac{\partial F}{\partial t}(t, x, q), \frac{\partial F}{\partial x}(t, x, q)).$$

We can easily show that Φ_F is a Legendrian unfolding associated with $(\pi_F, \tilde{\Phi}_F)$. By the same method of the theory of Arnol'd-Zakalyukin [1,12], we can show the following proposition.

PROPOSITION 3.3. *Every Legendrian unfolding germs are constructed by the above method.*

We now define a function germ

$$\tilde{F} : ((\mathbf{R}^{r-n} \times \mathbf{R}^n \times \mathbf{R}) \times \mathbf{R}^k, 0) \to (\mathbf{R}, 0)$$

by

$$\tilde{F}(t, x, y, q) = F(t, x, q) - y.$$

We call $\tilde{F}$ *a generating family* of Φ_F.

4. EQUIVALENCE RELATIONS

In order to study generic types of singularities appearing in solutions of equations, we will first introduce an equivalence relation among equation germs under the group of point transformations. *A point transformation* ϕ on $\mathbf{R}^n \times \mathbf{R}$ is, by definition, a diffeomorphism of $\mathbf{R}^n \times \mathbf{R}$ onto itself.

To define a lift of ϕ, we give a contact manifold which is a fiberwise compactification of $J^1(\mathbf{R}^n, \mathbf{R})$. Let

$$\tilde{\pi} : PT^*(\mathbf{R}^n \times \mathbf{R}) \to \mathbf{R}^n \times \mathbf{R}$$

be a projective cotangent bundle over $\mathbf{R}^n \times \mathbf{R}$. There exists a canonical contact structure on $PT^*(\mathbf{R}^n \times \mathbf{R})$ and $J^1(\mathbf{R}^n, \mathbf{R})$ is an affine part of it. Let $\phi : (\mathbf{R}^n \times \mathbf{R}, (x_0, y_0)) \to (\mathbf{R}^n \times \mathbf{R}, (x_1, y_1))$ be a diffeomorphism germ. Then we have a canonical contact lift

$$\hat{\phi} : (PT^*(\mathbf{R}^n \times \mathbf{R}), z_0) \to (PT^*(\mathbf{R}^n \times \mathbf{R}), z_1).$$

Following to Lie, the most natural equivalence relation among equation germs is given by point transformations. Let

$$f \text{ and } g : (\mathbf{R}^r, 0) \to J^1(\mathbf{R}^n, \mathbf{R})$$

be equation germs. We say that f and g are *equivalent as equations* if there exists a diffeomorphism germ

$$\psi : (\mathbf{R}^r, 0) \to (\mathbf{R}^r, 0)$$

and a point transformation

$$\phi : (\mathbf{R}^n \times \mathbf{R}, \pi(z_0)) \to (\mathbf{R}^n \times \mathbf{R}, \pi(z_1))$$

such that the lift $\hat{\phi}$ of ϕ satisfies the following

(1) $\hat{\phi}(z_0) = z_1$

(2) $\hat{\phi} \circ f = g \circ \psi$,

where $z_0 = f(0)$ and $z_1 = g(0)$.

Since we only treat completely integrable equations, we now introduce natural equivalence relation among equations with complete integral. Let

$$(\mu, f) \text{ and } (\mu', f') : (\mathbf{R}^r, 0) \to \mathbf{R}^n \times J^1(\mathbf{R}^n, \mathbf{R})$$

be equations with complete integral, where $\mu(0) = \mu'(0) = 0$, $f(0) = z_0$ and $f'(0) = z_1$. We say that (μ, f) and (μ', f') are *equivalent as equations with complete integral* if there exist diffeomorphism germs

$$\psi : (\mathbf{R}^r, 0) \to (\mathbf{R}^r, 0)$$

$$\kappa : (\mathbf{R}^{r-n}, 0) \to (\mathbf{R}^{r-n}, 0)$$

and a point transformation

$$\phi : (\mathbf{R}^n \times \mathbf{R}, \pi(z_0)) \to (\mathbf{R}^n \times \mathbf{R}, \pi(z_1))$$

such that the lift $\hat{\phi}$ of ϕ satisfies the following

(1) $\hat{\phi}(z_0) = z_1$

(2) $\hat{\phi} \circ f = f' \circ \psi$

(3) $\kappa \circ \mu = \mu' \circ \psi$.

It is clear that if (μ, f) and (μ', f') are equivalent as equations with complete integral then f and f' are equivalent as equations. The above definition leads us to the following equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be integral diagrams. We say that (μ, g) and (μ', g') are *equivalent* (respectively *strictly equivalent*) if the diagram

$$\begin{array}{ccccc} (\mathbf{R}^{r-n}, 0) & \xleftarrow{\mu} & (\mathbf{R}^r, 0) & \xrightarrow{g} & (\mathbf{R}^n \times \mathbf{R}, 0) \\ \kappa \downarrow & & \psi \downarrow & & \phi \downarrow \\ (\mathbf{R}^{r-n}, 0) & \xleftarrow{\mu'} & (\mathbf{R}^r, 0) & \xrightarrow{g'} & (\mathbf{R}^n \times \mathbf{R}, 0) \end{array}$$

commutes for some diffeomorphism germs κ, ψ and ϕ (respectively $\kappa = id_{\mathbf{R}^{r-n}}$).

Of course, if (μ, f) and (μ', f') are equivalent as equations with complete integral then $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent as integral diagrams. The converse is also true in the generic case.

THEOREM 4.1 ([10]). *Let*

$$(\mu, f) \text{ and } (\mu', f') : (\mathbf{R}^r, 0) \to (\mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}), 0 \times v)$$

be equations with complete integral such that regular points of $\pi \circ f | \mu^{-1}(t)$ and $\pi \circ f' | \mu'^{-1}(t)$ are respectively dense in $\mu^{-1}(t)$ and $\mu'^{-1}(t)$ for any $t \in (\mathbf{R}^{r-n}, 0)$. Then the followings are equivalent.

(1) *(μ, f) and (μ', f') are equivalent as equations with complete integral.*

(2) *$(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent as integral diagrams.*

In the case when $r = n + 1$ (the case of holonomic systems), we can assert that a more strict result.

THEOREM 4.2 ([10]). *Let*

$$(\mu, f) \text{ and } (\mu', f') : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R} \times J^1(\mathbf{R}^n, \mathbf{R}), 0 \times v)$$

be equations with complete integral such that the sets of critical points of $\pi \circ f$ *and* $\pi \circ f'$ *are closed sets without interior points. Then the followings are equivalent.*

(1) f *and* f' *are equivalent as equations.*

(2) (μ, f) *and* (μ', f') *are equivalent as equations with complete integral.*

(3) $(\mu, \pi \circ f)$ *and* $(\mu', \pi \circ f')$ *are equivalent as integral diagrams.*

In [10] we have developed a classification theory for integral diagram by some equivalence relations. Let (μ, g) and (μ', g') be integral diagrams. Then we say that (μ, g) and (μ', g') are *bifurcation equivalent (or, briefly $\mathcal{B}$-equivalent)* if the diagram

$$\begin{array}{ccccc} (\mathbf{R}^r, 0) & \xrightarrow{(\mu,g)} & (\mathbf{R}^{r-n} \times (\mathbf{R}^n \times \mathbf{R}), 0) & \xrightarrow{\pi_B} & (\mathbf{R}^{r-n}, 0) \\ \phi \downarrow & & \Psi \downarrow & & \downarrow \psi \\ (\mathbf{R}^r, 0) & \xrightarrow[(\mu',g')]{} & (\mathbf{R}^{r-n} \times (\mathbf{R}^n \times \mathbf{R}), 0) & \xrightarrow[\pi_B]{} & (\mathbf{R}^{r-n}, 0) \end{array}$$

commutes for some diffeomorphism germs ϕ, Ψ and ψ.

What does the $\mathcal{B}$-equivalence say? By the definition, if (μ, g) and (μ', g') are $\mathcal{B}$-equivalent, then $g|\mu^{-1}(t)$ and $g'|\mu'^{-1}(\psi(t))$ are right-left equivalent for any $t \in (\mathbf{R}^{r-n}, 0)$.

By the definition of complete integrals, $f|\mu^{-1}(t)$ are solutions of f for any $t \in (\mathbf{R}^{r-n}, 0)$. Thus the singularities of complete integrals are the Legendrian singularities of $f|\mu^{-1}(t)$. If $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are $\mathcal{B}$-equivalent, then bifurcations of Legendrian singularities of complete integrals along parameter $t \in (\mathbf{R}^{r-n}, 0)$ are diffeomorphic. Of course, if (μ, g) and (μ', g') are equivalent then these are $\mathcal{B}$-equivalent. But, in the case when $r > n+1$, the classification by the equivalence is very hard. By the method of Legendre transformations, we have an algorithm to classify integral diagrams by the $\mathcal{B}$-equivalence. The list of generic classification in the case where $n = 1, 2$ and $r = 2, 3, 4.$ is as follows.

THEOREM 4.3 ([10]). A) *For generic equation germ with complete integral*

$$(\mu, f) : (\mathbf{R}^2, 0) \to \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}),$$

the integral diagram $(\mu, \pi \circ f)$ *is* $\mathcal{B}$*-equivalent to one of the following list :*

(1) $\mu = u_2$, $g = (u_1, u_2)$

(2) $\mu = u_2$, $g = (3u_1^2, 2u_1^3 - u_2)$

(3) $\mu = u_2$, $g = (4u_1^3 + 2u_2(u_1 + 1), 3u_1^4 + u_2 u_1^2)$.

B) *For generic equation germ with complete integral*

$$(\mu, f) : (\mathbf{R}^3, 0) \to \mathbf{R} \times J^1(\mathbf{R}^2, \mathbf{R}),$$

the integral diagram $(\mu, \pi \circ f)$ *is* $\mathcal{B}$*-equivalent to one of the following list :*

(1) $\mu = u_3$, $g = (u_1, u_2, u_3 + u_1^2 + u_2^2)$

(2) $\mu = u_3$, $g = (u_1^2, u_2, 2u_1^3 + u_2^2 - u_3)$
(3) $\mu = u_3$, $g = (u_1^3 + u_1u_2, u_2, 3u_1^4 + u_2^2 + 2u_1^2u_2 - u_3)$
(4) $\mu = u_3$, $g = (5u_1^4 + (3u_1^2 + 1)u_3 + 4u_1u_2, u_2, 4u_1^5 + u_2^2 + 2u_1^3u_3 + 2u_1^2u_2)$
(5) $\mu = u_3$, $g = (u_1u_2, u_1^2 \pm 3u_2^2 + 2u_2u_3, 2u_1^2u_2 \pm 2u_2^3 + u_2^2u_3 - u_3)$.

C) *For generic equation germ with complete integral*

$$(\mu, f) : (\mathbf{R}^4, 0) \to \mathbf{R}^2 \times J^1(\mathbf{R}^2, \mathbf{R}),$$

the integral diagram $(\mu, \pi \circ f)$ *is* $\mathcal{B}$*-equivalent to one of the following list :*
(1) $\mu = (u_3, u_4)$, $g = (u_1, u_2, u_1^2 + u_2^2 - u_3 - u_4)$
(2) $\mu = (u_3, u_4)$, $g = (u_1^2, u_2, 2u_1^3 + u_2^2 - u_3 - u_4)$
(3) $\mu = (u_3, u_4)$, $g = (u_1^3 + u_1u_2, u_2, 3u_1^4 + u_2^2 + 2u_1^2u_2 - u_3 - u_4)$
(4) $\mu = (u_3, u_4)$, $g = (5u_1^4 + (3u_1^2 + 1)u_3 + 4u_1u_2, u_2, 4u_1^5 + u_2^2 + 2u_1^3u_3 + 2u_1^2u_2 + u_4)$
(5) $\mu = (u_3, u_4)$, $g = (u_1u_2, u_1^2 \pm 3u_2^2 + 2u_2u_3, 2u_1^2u_2 \pm 2u_2^3 + u_2^2u_3 - u_3 - u_4)$.

By the aid of Theorem 4.3, we can easily draw pictures of singularities of solutions. We used Mathematica on MacIIci in order to draw pictures. Fig. 3 is a picture of bifurcations of singularities of solutions of type A), 3).

```
Do[ParametricPlot[{4*t^3+2*n*(t+1),5*(3*t^4+n*t^2)},{t,-2,2},
     PlotRange->{-10,10}],{n,-2,2,2}]
```

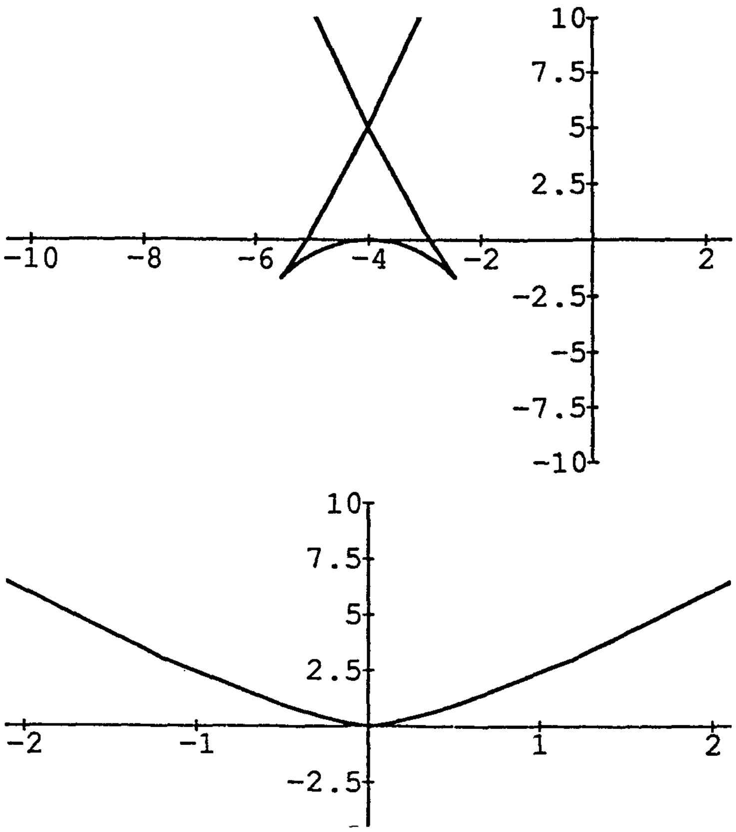


Fig. 3 (i) (ii)

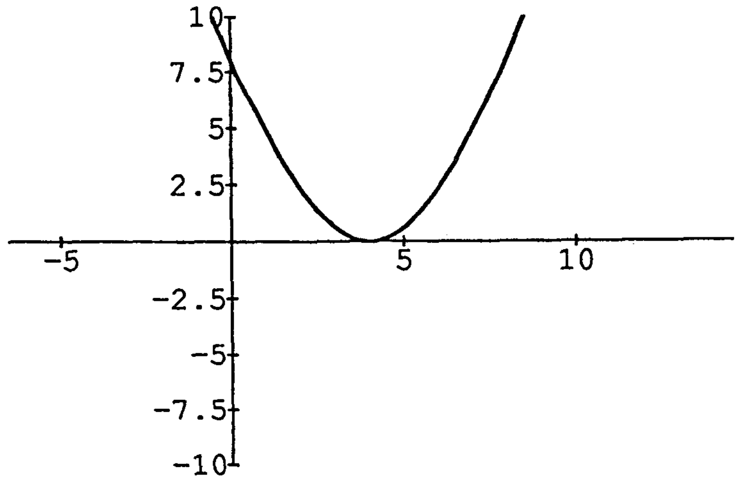


Fig. 3 (iii)

```
ParametricPlot3D[
{u^3+u*v,v,3*u^4+v^2+2*u^2*v},
{u,-1.2,1.2},{v,-1.5,0.5},
      BoxRatios->{-5,5,2}]
```

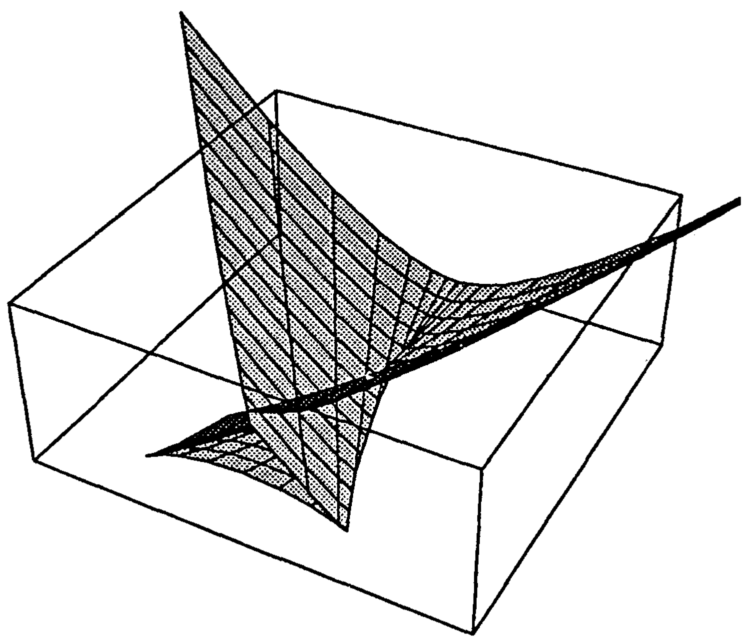


Fig. 4

Fig. 4 is a picture of the type B), 3).

On the other hand, in the case when $r = n + 1$ (the case of holonomic systems), we have results about the strictly equivalence. In [8] we have classified in the case when $n = 1$ (the case of ordinary differential equations).

THEOREM 4.4 ([8], THEOREM B). *For generic equation germ with complete integral*

$$(\mu, f) : (\mathbf{R}^2, 0) \to \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}),$$

the integral diagram $(\mu, \pi \circ f)$ *is strictly equivalent to one of germs in the following list :*

(1) $\mu = u_2$, $g = (u_1, u_2)$,

(2) $\mu = u_2 - \frac{2}{3}u_1^3$, $g = (u_1^2, u_2)$,

(3) $\mu = u_2 - \frac{1}{2}u_1$, $g = (u_1, u_2^2)$,

(4) $\mu = \frac{3}{4}u_1^4 + \frac{1}{2}u_1^2 u_2 + \alpha \circ g$, $g = (u_1^3 + u_1 u_2, u_2)$, *where* $\alpha(x, y)$ *is a* C^∞ *function germ with* $\alpha(0) = 0$, *and* $\frac{\partial \alpha}{\partial y}(0) = \pm 1$,

(5) $\mu = u_2 + \alpha \circ g$, $g = (u_1, u_2^3 + u_1 u_2)$, *where* $\alpha(x, y)$ *is a* C^∞ *function germ with* $\alpha(0) = 0$,

(6) $\mu = \frac{1}{2}u_2^2 + \alpha \circ g$, $g = (u_1, u_2^3 + u_1 u_2^2)$, *where* $\alpha(x, y)$ *is a* C^∞ *function germ with* $\alpha(0) = 0$ and $\frac{\partial \alpha}{\partial x}(0) = 1$.

We remark that α in the above theorem is a 'functional moduli'relative to the strict equivalence. The method of the classification of the above theorem is rather a complicated. At first, we have decided the normal form of $\pi \circ f$ by the right-left equivalence for generic (μ, f). And we applied $\pi \circ f$-compatible vector fields to detect the normal form of μ. This method is analogous to that of the classification of divergent diagrams in Dufour [5]. But it seems to be very hard to generalize this method to the case when $n \geq 2$. By the aid of generating families, we will give another method to classify integral diagrams $(\mu, \pi \circ f)$ by the strict equivalence in the forthcoming paper [10] which is easier than the above method. One of the results is the classification in the case when $n = 2$.

THEOREM 4.5 ([10]). *For generic equation germ with complete integral*

$$(\mu, f) : (\mathbf{R}^3, 0) \to \mathbf{R} \times J^1(\mathbf{R}^2, \mathbf{R}),$$

the integral diagram $(\mu, \pi \circ f)$*is strictly equivalent to one of germs in the following list :*

(1) $\mu = u_3$, $g = (u_1, u_2, u_3)$,

(2) $\mu = \frac{2}{3}u_1^3 + \frac{1}{2}u_2 u_1^2 + u_2$, $g = (u_1^2, u_2, u_3)$,

(3) $\mu = u_3 + \frac{1}{2}u_1$, $g = (u_1, u_2, u_3^2)$,

(4) $\mu = \frac{3}{4}u_1^4 + \frac{1}{2}u_1^2 u_2 + u_3$, $g = (u_1^3 + u_1 u_2, u_2, u_3)$,

(5) $\mu = u_3$, $g = (u_1, u_2, u_3^3 + u_2 u_3)$,

(6) $\mu = \frac{1}{2}u_3^2 + \frac{2}{3}u_3 u_1$, $g = (u_1, u_2, u_3^3 + u_3^2 u_1,)$

(7) $\mu = \frac{4}{5}u_1^5 + \frac{1}{2}u_2 u_1^2 + \frac{2}{3}u_3 u_1^3 + u_3 + \alpha \circ g$, $g = (u_1^4 + u_2 u_1 + u_3 u_1^2, u_2, u_3)$,

(8) $\mu = \pm(2u_2^3 - 3u_2^2 \mp u_3^2 - 2u_1 u_2 + u_1 u_2^2 \pm 2u_2 u_3^2 + u_1) + \alpha \circ g$, $g = (u_1, u_2 u_3, \pm 3u_2^2 + u_3^2 \pm 2u_1 u_2)$,

(9) $\mu = u_3 + \alpha \circ g$, $g = (u_1, 3u_3^2 + u_1 u_2, u_3^2 + 2u_2^3)$,

(10) $\mu = u_3 + \alpha \circ g$, $g = (u_1, u_2, u_3^4 + u_1 u_3^2 + u_2 u_3)$,

(11) $\mu = 4u_3^3 + 3u_1 u_3^2 + 2u_2 u_3^3 + u_1 + \alpha \circ g$, $g = (u_1, u_2, 3u_3^4 + 2u_1 u_3^3 + u_2 u_3^2)$.

REFERENCES

[1]. V.I. Arnol'd, S.M. Gusein-Zade and A.N. Varchenko, "Singularities of differentiable maps, vol. 1, Monographs in Math. 82," Birkhauser, 1985.

[2]. C. Caratheodory, "Calculous of Variations and Partial Differential Equations of First Order, Part I, Partial Differential Equations of the First Order," Holden-Day, 1965.

[3]. L. Dara, *Singularités génériqu des équations differentielles multiformes*, Bol. Soc. Brasil Mat. (1975), 95–128.

[4]. M.G. Darboux, *Sur les solutions singuli`res des équations aux derivees ordinaries du premier order*, Bull. Sciences Math. and Astron. (1873), 158–176.

[5]. J.P. Dufour, *Families de courbes planes différentiables*, Topology (1983), 449–474.

[6]. W. Dyck, *Über die Gestalich Verhaltinisse der durch eine Differentialgleiehung erster Ordung zweischen zwei Variablen definiren Curven Systeme*, Bay. Akad. Der Wissenschaften München Sizungberichte der Math. Phys. Classe. (1891), 23–57.

[7]. M. Fukuda and T. Fukuda, *Singular solutions of ordinary differential equations*, The Yokohama Math. Jour. (1977), 41–58.

[8]. A. Hayakawa, G. Ishikawa, S.Izumiya and K. Yamaguchi, *Classification of generic integral diagram and first order ordinary differential equations*, preprint.

[9]. S. Izumiya, *Generic properties of first order partial differential equations*, preprint.

[10]. S. Izumiya, *Theory of Legendrian unfoldings and first order differential equations*, in preparation.

[11]. R. Thom, *Sur les equations différentielles multiformes et leurs integrals singulieres*, Bol. da Soc. Brasil Mat. (1972), 1–11.

[12]. V.M. Zakalyukin, *Lagrangian and Legendrian singularites*, Funct. Anal. Appl. (1976), 23–31.

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